

# A RECURSION FORMULA FOR MOMENTS OF DERIVATIVES OF RANDOM MATRIX POLYNOMIALS

S. ALİ ALTUĞ, SANDRO BETTIN, IAN PETROW, RISHIKESH, AND IAN WHITEHEAD

**ABSTRACT.** We give asymptotic formulae for random matrix averages of derivatives of characteristic polynomials over the groups  $\mathrm{USp}(2N)$ ,  $\mathrm{SO}(2N)$  and  $\mathrm{O}^-(2N)$ . These averages are used to predict the asymptotic formulae for moments of derivatives of  $L$ -functions which arise in number theory. Each formula gives the leading constant of the asymptotic in terms of determinants of hypergeometric functions. We find a differential recurrence relation between these determinants which allows the rapid computation of the  $(k+1)$ -st constant in terms of the  $k$ -th and  $(k-1)$ -st. This recurrence is reminiscent of a Toda lattice equation arising in the theory of  $\tau$ -functions associated with Painlevé differential equations.

## 1. INTRODUCTION

For over 50 years, mathematicians and physicists have used random matrix theory to study a wide-ranging, growing list of probabilistic phenomena. Particularly surprising are its applications in number theory, where random matrices model the distribution of nontrivial zeros of the Riemann zeta function. Random matrix theory now provides far-reaching and widely believed conjectures for many questions in the analytic theory of  $L$ -functions.

Katz and Sarnak [KaSa] give evidence that every family of  $L$ -functions falls into one of four symmetry types: unitary  $\mathrm{U}(N)$ , unitary symplectic  $\mathrm{USp}(2N)$ , even orthogonal  $\mathrm{SO}(2N)$  and odd orthogonal  $\mathrm{O}^-(2N)$ . These symmetry types govern the distribution of zeroes and special values in families. Using random matrix models, Keating and Snaith [KeSn] and Conrey, Farmer, Keating, Rubinstein and Snaith [CFKRS03] have produced deep conjectures for estimating the integral moments of central values in families of  $L$ -functions.

The derivatives of  $L$ -functions are also of great interest, and are the subject of this paper. A motivational example is Speiser's theorem, which asserts that the Riemann hypothesis is equivalent to the nonexistence of nonreal zeros of the derivative of the Riemann zeta function to the left of the critical line, see e.g. [Sou]. Moreover, the derivatives of  $L$ -functions control the order of vanishing at the central point, which encodes important arithmetic and geometric information. For example, according to the Birch and Swinnerton-Dyer Conjecture, the order of vanishing of the  $L$ -function of an elliptic curve over the rationals coincides with the arithmetic rank of the curve.

An  $L$ -function is modeled by the characteristic polynomial  $\Lambda_A$  of a random matrix  $A$ . Here we compute

$$M_k(\mathrm{G}(2N), m) := \int_{\mathrm{G}(2N)} \left( \Lambda_A^{(m)}(1) \right)^k dA,$$

where  $\mathrm{G}$  denotes  $\mathrm{USp}$ ,  $\mathrm{SO}$ , or  $\mathrm{O}^-$ , and  $dA$  is the Haar measure on  $\mathrm{G}$ . As  $N \rightarrow \infty$ , this models the  $k$ th moment of  $L^{(m)}(1/2)$  in a family of symmetry type  $\mathrm{G}$ .

One can find the moments of  $\Lambda_A^{(m)}(1)$  by differentiating the corresponding shifted moment formulae, which are computed in [CFKRS03]. Conrey, Rubinstein and Snaith in [CRS] develop a faster method to compute the relevant averages in the unitary case:

$$\int_{U(N)} |\Lambda'_A(1)|^{2k} dA = b_k N^{k^2+2k} + O\left(N^{k^2+2k-1}\right).$$

This leading constant  $b_k$  is the same “geometric constant” appearing in conjectures for the asymptotic estimate of the  $k$ -th moment of  $\zeta'(s)$  in  $t$  aspect. Conrey, Rubinstein and Snaith describe  $b_k$  in terms of a  $k \times k$  determinant of  $I$ -Bessel functions, and are able to compute  $b_k$  numerically for  $k \leq 15$ .

Forrester and Witte in [FW06,FW02] find a surprising expression for these determinants of  $I$ -Bessel functions in terms of solutions to Painlevé III' differential equations. Explicitly, the formula in [CRS] is as follows:

$$b_k = (-1)^k \sum_{h=0}^k \binom{k}{h} \left(\frac{d}{dt}\right)^{h+k} \left( e^{-t} t^{-k^2/2} \det_{k \times k} \left( I_{k+i-j}(2\sqrt{t}) \right) \right) \Big|_{t=0}$$

where  $I_\ell(x)$  is the modified Bessel function of the first kind. One then defines

$$\tau_k(t) := 2^{-k(k-1)} t^{-k^2/2} \det_{k \times k} \left( I_{k+i-j}(2\sqrt{t}) \right).$$

Forrester and Witte [FW06,FW02] find that this  $\tau_k(t)$  (denoted  $\tau[k](t)$  in [FW02, section 4]) is in fact the Okamoto  $\tau$  function associated with the Painlevé III' differential equation:

$$(ty'')^2 + y'(4y' - 1)(y - ty') - \frac{1}{4}k^2 = 0.$$

This nonlinear second order differential equation has a solution with certain boundary data (see [FW02]) given in terms of  $\tau_k(t)$  by the formula

$$y = \sigma_{\text{III},k}(t) = -t \frac{d}{dt} \log \left( e^{-t/4} t^{k^2} \tau_k \left( \frac{t}{4} \right) \right).$$

Specifying boundary conditions, one can quickly compute  $\sigma_{\text{III},k}(t)$  from the differential equation and recover  $\tau_k(t)$  via the equation

$$\tau_k(t) = \exp \left( - \int_0^{4t} \left( \sigma_{\text{III},k}(s) + k^2 - \frac{s}{4} \right) \frac{ds}{s} \right).$$

This expression allows a much faster computation of the constants  $b_k$ .

The goal of the present paper is to extend these results to the other symmetry types relevant to  $L$ -functions:  $\text{USp}(2N)$ ,  $\text{SO}(2N)$  and  $\text{O}^-(2N)$ .

Employing similar techniques to [CRS], we obtain analogous results, where the role of the  $I$ -Bessel functions above is here played by hypergeometric functions,

$$\begin{aligned} g_m(u) &= \frac{1}{2\pi i} \oint_{|w|=1} \frac{e^{w+\frac{u}{w^2}}}{w^{m+1}} dw \\ &= \frac{1}{\Gamma(m+1)} {}_0F_2 \left( ; \frac{m}{2} + 1, \frac{m+1}{2}; \frac{u}{4} \right), \end{aligned}$$

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for  $u \in \mathbb{C}$  and  $m \in \mathbb{Z}$ . For negative  $m$ , interpret the above expression as the limit. The role of the  $\tau$ -function is played by

$$\mathcal{T}_{k,\ell}(u) := \det_{k \times k} (g_{2i-j+\ell}(u)), \quad (1)$$

for  $k \geq 0$ ,  $\ell \in \mathbb{Z}$  and  $u \in \mathbb{C}$ , where, here and in the following, the indices  $i$  and  $j$  of the matrix in the determinant range from 1 to  $k$ . In the context of Theorems 1, 2 and 3 below, the  $\ell$  appearing here takes values 0,  $-1$ , 0 respectively. We now state theorems for the symplectic, special orthogonal, and negative orthogonal cases:

**Theorem 1.** *We have*

$$M_k(\mathrm{USp}(2N), 2) = b_k(\mathrm{USp}(2N), 2) \cdot (2N)^{\frac{k^2+5k}{2}} + O(N^{\frac{k^2+3k}{2}}) \quad (2)$$

where

$$b_k(\mathrm{USp}(2N), 2) = 2^{-\frac{k^2+5k}{2}} \frac{d^k}{du^k} (e^u \mathcal{T}_{k,0}(2u))|_{u=0}.$$

**Theorem 2.** *We have*

$$M_k(\mathrm{SO}(2N), 2) = b_k(\mathrm{SO}(2N), 2) \cdot (2N)^{\frac{k^2+3k}{2}} + O(N^{\frac{k^2+k}{2}}) \quad (3)$$

where

$$b_k(\mathrm{SO}(2N), 2) = 2^{-\frac{k^2+k}{2}} \frac{d^k}{du^k} (e^u \mathcal{T}_{k,-1}(2u))|_{u=0}.$$

**Theorem 3.** *We have*

$$M_k(\mathrm{O}^-(2N), 3) = b_k(\mathrm{O}^-(2N), 3) \cdot (2N)^{\frac{k^2+5k}{2}} + O(N^{\frac{k^2+3k}{2}}) \quad (4)$$

where

$$b_k(\mathrm{O}^-(2N), 3) = 3 \cdot 2^{-\frac{k^2+3k}{2}} \frac{d^k}{du^k} (e^u \mathcal{T}_{k,0}(2u))|_{u=0}.$$

Note that we find above that  $b_k(\mathrm{O}^-(2N), 2) = 3 \cdot 2^k \cdot b_k(\mathrm{USp}(2N), 2)$ .

We are naturally led to consider the second and third derivatives of characteristic polynomials in the above theorems instead of the first derivative due to root number considerations. Indeed, if  $A$  is a unitary matrix, the characteristic polynomial satisfies the functional equation

$$\Lambda_A(s) = (-s)^N (\det A)^{-1} \overline{\Lambda_A(s^{-1})},$$

where  $\overline{f}(s) = \overline{f(\overline{s})}$ . When  $A$  is in  $\mathrm{USp}(2N)$  or  $\mathrm{SO}(2N)$  then  $\det A$  is constantly equal to  $+1$  and one has a simple expression for  $\Lambda'_A(1)$  in terms of  $\Lambda_A(1)$ , which can be used to compute the moments of the derivative via partial integration. Thus the moments of  $\Lambda''_A(1)$  give the next novel information. When  $A \in \mathrm{O}^-(2N)$  we have  $\det A = -1$ , and thus  $\Lambda_A(1) = 0$ . In this case  $\Lambda'_A(1)$  plays the role which  $\Lambda_A(1)$  plays in the other families,  $\Lambda''_A(1)$  has a simple expression in terms of  $\Lambda'_A(1)$ , and therefore one considers moments of  $\Lambda'''_A(1)$ . The same reasoning carries over to the families of  $L$ -functions having each of the aforementioned symmetry types.

The method of proof of Theorems 1, 2 and 3 can be generalized to higher-order derivatives easily. It suffices to expand the binomial in Lemma 4, and otherwise proceed as in the given proofs of Theorems 1, 2 and 3.

We can use Theorems 1, 2 and 3 to give conjectures for moments of derivatives of  $L$ -functions at  $s = 1/2$  with the above symmetry types. For example, the quadratic Dirichlet  $L$ -functions ordered by conductor form a symplectic family and thus we make the following conjecture:

**Conjecture 1.** *Let  $\mathcal{D}(X) := \{|d| < X, d \text{ fundamental discriminant}\}$ , and  $L(s, \chi_d)$  denote the quadratic Dirichlet  $L$ -function of fundamental discriminant  $d$ . The average value of the second derivative of quadratic Dirichlet  $L$ -functions at the central point is*

$$\frac{1}{|\mathcal{D}(X)|} \sum_{d \in \mathcal{D}(X)} L''(1/2, \chi_d)^k \sim a_k \cdot b_k(\mathrm{USp}(2N), 2) \cdot (\log X)^{\frac{k^2+5k}{2}}$$

Here  $a_k$  is a well understood arithmetic constant depending on the functional equations in the family:

$$a_k = \prod_{p \text{ prime}} \frac{(1 - 1/p)^{k(k+1)/2}}{1 + 1/p} \left( \frac{(1 - 1/\sqrt{p})^{-k} + (1 + 1/\sqrt{p})^{-k}}{2} + \frac{1}{p} \right)$$

(see, for example [CFKRS05, 1.3.5]). This is the same arithmetic constant appearing in moment conjectures for  $L(1/2, \chi_d)$  without the derivative. It is not predicted by random matrix calculations.

Our Theorems 1, 2 and 3 along with the results of [CRS] allow similar conjectures to be made for any family of  $L$ -functions.

Under closer examination, the determinants  $\mathcal{T}_{k,\ell}(u)$  defined by (1) and appearing in Theorems 1, 2 and 3 exhibit a surprisingly rich structure. Our Theorem 4 is a differential recurrence relation which allows much faster computation of the constants  $b_k(\mathrm{G}(N), m)$ :

**Theorem 4.** *Let  $k \in \mathbb{Z}_{>0}$ ,  $\ell \in \mathbb{Z}$ . Then*

$$\mathcal{T}_{k+1,\ell}(u) \mathcal{T}_{k-1,\ell}(u) = 2 \left( u \mathcal{T}_{k,\ell}(u) \mathcal{T}_{k,\ell}''(u) + \mathcal{T}_{k,\ell}(u) \mathcal{T}_{k,\ell}'(u) - u (\mathcal{T}_{k,\ell}'(u))^2 \right). \quad (5)$$

This recurrence relation closely resembles a Toda lattice equation for the Okamoto  $\tau$ -function associated with a Painlevé differential equation, see [Oka, Theorem 2]. Such Toda lattice equations are at the heart of the  $\tau$ -function theory of Painlevé equations, and are used by Forrester and Witte [FW06, FW02] to connect determinants of  $I$ -Bessel functions found by [CRS] to the Painlevé III' equation. It would be very interesting to determine whether or not there exists a differential equation arising from our formula (5) which plays the role for symplectic and orthogonal types that Painlevé III' plays for unitary symmetry.

Ultimately, one hopes to obtain formulae for the complex moments of characteristic polynomials in the various symmetry types. In the case of the undifferentiated moment conjectures, it has been found that the geometric constants  $g_k$  can be expressed in a simple form in terms of Barnes  $G$ -functions, which are well-defined for complex values of  $k$ , see [CF]. A project for the future would be to see if there exists a similar expression for the geometric coefficients  $b_k$  studied in this paper for moments of derivatives of  $L$ -functions.

By computing the expressions in Theorems 1, 2 and 3 directly, one can obtain the values of  $b_k$  up to  $k \approx 10$ . By using Theorem 4 we do much better: running SAGE for about an hour on a machine with 4 gigabytes of RAM we computed the first 200 values of  $b_k(\mathrm{USp}(2N), 2)$ . In section 4 we give a table with the first 10 values of  $b_k$  for each symmetry type. To give

an example, we have

$$b_{10}(\mathrm{USp}(2N)) = \frac{47 \cdot 1553 \cdot 1787 \cdot 73709 \cdot 152825093}{2^{62} \cdot 3^{34} \cdot 5^{17} \cdot 7^{10} \cdot 11^5 \cdot 13^5 \cdot 17^4 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37},$$

$$b_{10}(\mathrm{SO}(2N)) = \frac{25171 \cdot 7695491 \cdot 57668937071891}{2^{45} \cdot 3^{29} \cdot 5^{15} \cdot 7^9 \cdot 11^5 \cdot 13^5 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37},$$

$$b_{10}(\mathrm{O}^-(2N)) = \frac{47 \cdot 1553 \cdot 1787 \cdot 73709 \cdot 152825093}{2^{52} \cdot 3^{33} \cdot 5^{17} \cdot 7^{10} \cdot 11^5 \cdot 13^5 \cdot 17^4 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37}.$$

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## 2. SOME LEMMAS

We recall the definitions of the relevant spaces of matrices:  $\mathrm{USp}(2N)$  is the subgroup of  $2N \times 2N$  unitary matrices  $M$  with  $M^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ , where  $I$  denotes the  $N \times N$  identity matrix.  $\mathrm{SO}(2N)$  and  $\mathrm{O}^-(2N)$  are the subsets ( $\mathrm{O}^-$  is not a subgroup) of orthogonal matrices with determinant 1 and  $-1$ , respectively. These compact spaces admit Haar measures which we normalize so that the volume of each space is 1. A matrix in  $\mathrm{USp}(2N)$  or  $\mathrm{SO}(2N)$  has characteristic polynomial of the form  $\Lambda(x) = \prod_{n=1}^N (1 - e^{i\theta_n} x)(1 - e^{-i\theta_n} x)$ , and a matrix in  $\mathrm{O}^-(2N)$  has characteristic polynomial of the form  $\Lambda(x) = (1 - x)(1 + x) \prod_{n=1}^{N-1} (1 - e^{i\theta_n} x)(1 - e^{-i\theta_n} x)$ , with  $\theta_n \in \mathbb{R}$ .

The shifted moments of  $\Lambda$  are defined as follows:

$$I(\mathrm{G}(2N); z_1, \dots, z_k) := \int_{\mathrm{G}(2N)} \Lambda(z_1) \cdots \Lambda(z_k) dA.$$

Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS03] use Weyl integration formula (see, for example, Theorem 8.60 of [Kn]) to compute the following shifted moment formulae, which are the starting point for our work (note that we have corrected a typo in [CFKRS03, 4.9]):

**Lemma 1.** [CFKRS03, 3.36] Assume that  $\alpha_1, \dots, \alpha_k$  are complex numbers with  $|\alpha_i| < 1$  for  $i = 1, \dots, k$ . Then

$$\begin{aligned} I(\mathrm{USp}(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) &= \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint_{|w_1|=1} \cdots \oint_{|w_k|=1} \frac{\prod_{1 \leq i < j \leq k} (w_i^2 - w_j^2)^2 \prod_{j=1}^k w_j}{\prod_{1 \leq i, j \leq k} (w_j^2 - \alpha_i^2)} \times \\ &\quad \times e^{N \sum_{j=1}^k (w_j - \alpha_j)} \prod_{1 \leq m \leq \ell \leq k} (1 - e^{-w_m - w_\ell})^{-1} dw_1 \cdots dw_k. \end{aligned}$$

**Lemma 2.** [CFKRS03, 4.43] Assume that  $\alpha_1, \dots, \alpha_k$  are complex numbers with  $|\alpha_i| < 1$  for  $i = 1, \dots, k$ . Then

$$\begin{aligned} I(\mathrm{SO}(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) &= \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint_{|w_1|=1} \dots \oint_{|w_k|=1} \frac{\prod_{1 \leq i < j \leq k} (w_i^2 - w_j^2)^2 \prod_{j=1}^k w_j}{\prod_{1 \leq i, j \leq k} (w_j^2 - \alpha_i^2)} \times \\ &\quad \times e^{N \sum_{j=1}^k (w_j + \alpha_j)} \prod_{1 \leq m < \ell < k} (1 - e^{-w_m - w_\ell})^{-1} dw_1 \dots dw_k. \end{aligned}$$

**Lemma 3.** [CFKRS03, 4.9] Assume that  $\alpha_1, \dots, \alpha_k$  are complex numbers with  $|\alpha_i| < 1$  for  $i = 1, \dots, k$ . Then

$$\begin{aligned} I(\mathrm{O}^-(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) &= \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint_{|w_1|=1} \dots \oint_{|w_k|=1} \frac{\prod_{1 \leq i < j \leq k} (w_i^2 - w_j^2)^2 \prod_{j=1}^k \alpha_j}{\prod_{1 \leq i, j \leq k} (w_j^2 - \alpha_i^2)} \times \\ &\quad \times e^{N \sum_{j=1}^k (w_j + \alpha_j)} \prod_{1 \leq m \leq \ell \leq k} (1 - e^{-w_m - w_\ell})^{-1} dw_1 \dots dw_k. \end{aligned}$$

In fact, we will use approximate versions of the lemmas above, which follow immediately from the fact that  $(1 - e^{-x})^{-1} = x^{-1} + O(1)$ . To simplify notation, we denote Vandermonde determinants as follows:

$$\Delta(w) := \det_{k \times k} (w_i^{j-1}) = \prod_{1 \leq i < j \leq k} (w_i - w_j)$$

and we write  $w^2 = (w_i^2)_{1 \leq i \leq k}$  for any  $w = (w_i)_{1 \leq i \leq k} \in \mathbb{C}^k$ . Then we have:

**Corollary 1.** Assume that  $\alpha_1, \dots, \alpha_k$  are complex numbers such that  $|\alpha_j| \ll \frac{1}{N}$  for  $j = 1 \dots k$ . Then

$$\begin{aligned} I(\mathrm{USp}(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) &= \\ &= \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k k!} \left( \oint \dots \oint \frac{\Delta(w) \Delta(w^2) e^{N \sum_j (w_j - \alpha_j)}}{\prod_{i,j} (w_j^2 - \alpha_i^2)} dw_1 \dots dw_k \right) (1 + O(N^{-1})). \end{aligned}$$

**Corollary 2.** Assume that  $\alpha_1, \dots, \alpha_k$  are complex numbers such that  $|\alpha_j| \ll \frac{1}{N}$  for  $j = 1 \dots k$ . Then

$$\begin{aligned} I(\mathrm{SO}(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) &= \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \left( \oint \dots \oint \frac{\Delta(w) \Delta(w^2) (\prod_j w_j) e^{N \sum_j (w_j + \alpha_j)}}{\prod_{i,j} (w_j^2 - \alpha_i^2)} dw_1 \dots dw_k \right) (1 + O(N^{-1})). \end{aligned}$$

**Corollary 3.** Assume that  $\alpha_1, \dots, \alpha_k$  are complex numbers such that  $|\alpha_j| \ll \frac{1}{N}$  for  $j = 1 \dots k$ . Then

$$\begin{aligned} I(\mathrm{O}^-(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) &= \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \left( \oint \dots \oint \frac{\Delta(w) \Delta(w^2) (\prod_j \alpha_j) e^{N \sum_j (w_j + \alpha_j)}}{\prod_{i,j} (w_j^2 - \alpha_i^2)} dw_1 \dots dw_k \right) (1 + O(N^{-1})). \end{aligned}$$

The following formula will appear in the proofs of Theorems 1 and 2, below. We use it only when  $m = 2$ , but the general statement is a starting point for computing moments of the  $m$ th derivative.

**Lemma 4.** *For  $m \geq 0$ , we have*

$$\begin{aligned} \frac{d^m}{d\alpha_1^m} \cdots \frac{d^m}{d\alpha_k^m} \frac{e^{-N \sum_{i=1}^k \alpha_i}}{\prod_{1 \leq i, j \leq k} (w_j^2 - \alpha_i^2)} \Big|_{\alpha_1 = \dots = \alpha_k = 0} &= \\ &= \left( \sum_{\ell=0}^m \binom{m}{\ell} (-N)^{m-\ell} \sum_{\substack{i_1 + \dots + i_k = \ell, \\ i_j \text{ even}}} \prod_{j=1}^k \frac{i_j!}{w_j^{i_j+2}} \right)^k \end{aligned}$$

*Proof.* We have

$$\frac{d^m}{d\alpha^m} \frac{e^{-N\alpha}}{\prod_{1 \leq j \leq k} (w_j^2 - \alpha^2)} = \sum_{\ell=0}^m \binom{m}{\ell} (-N)^{m-\ell} e^{-N\alpha} \sum_{i_1 + \dots + i_k = \ell} \prod_{j=1}^k \frac{d^{i_j}}{d\alpha^{i_j}} \frac{1}{(w_j^2 - \alpha^2)}$$

and

$$\frac{d^q}{d\alpha^q} \left( \frac{1}{w^2 - \alpha^2} \right) \Big|_{\alpha=0} = \begin{cases} 0 & \text{if } q \text{ is odd} \\ q! / w^{q+2} & \text{if } q \text{ is even.} \end{cases}$$

Therefore,

$$\frac{d^m}{d\alpha^m} \frac{e^{-N\alpha}}{\prod_{1 \leq j \leq k} (w_j^2 - \alpha^2)} \Big|_{\alpha=0} = \sum_{\ell=0}^m \binom{m}{\ell} (-N)^{m-\ell} e^{-N\alpha} \sum_{\substack{i_1 + \dots + i_k = \ell, \\ i_j \text{ even}}} \prod_{j=1}^k \frac{i_j!}{w_j^{i_j+2}}$$

and the lemma follows.  $\square$

Our proofs will also rely upon Vandermonde determinants of differential operators:

$$\Delta \left( \frac{d}{dx} \right) := \prod_{1 \leq i < j \leq k} \left( \frac{d}{dx_i} - \frac{d}{dx_j} \right) = \det_{k \times k} \left( \frac{d^{j-1}}{dx_i^{j-1}} \right).$$

We give two lemmas on computing with these—Lemma 5 is a direct consequence of the definition, but we prove Lemma 6 in detail.

**Lemma 5.** *Let  $f_1(x), \dots, f_k(x)$  be  $k-1$  times differentiable. Then*

$$\Delta \left( \frac{d}{dx} \right) \prod_{i=1}^k f_i(x_i) = \det_{k \times k} \left( f_i^{(j-1)}(x_i) \right).$$

**Lemma 6.** *Let  $f(x, y)$  be  $k-1$  times differentiable in  $x$  and  $y$ . Then*

$$\Delta \left( \frac{d}{dx} \right) \Delta \left( \frac{d}{dy} \right) \prod_{i=1}^k f(x_i, y_i) \Big|_{\substack{x_1 = \dots = x_k = X, \\ y_1 = \dots = y_k = Y}} = k! \det_{k \times k} \left( \frac{d^{i+j-2}}{dX^{i-1} dY^{j-1}} f(X, Y) \right)$$

*Proof.* By Lemma 5 we have

$$\Delta \left( \frac{d}{dx} \right) \prod_{i=1}^k f(x_i, y_i) = \det_{k \times k} \left( \frac{d^{j-1}}{dx_i^{j-1}} f(x_i, y_i) \right) = \sum_{\mu} \text{sign}(\mu) \prod_{i=1}^k \frac{d^{\mu(i)-1}}{dx_i^{\mu(i)-1}} f(x_i, y_i)$$

where the sum runs over the permutations  $\mu \in S_k$ . Applying Lemma 5 again, we find

$$\begin{aligned} \Delta \left( \frac{d}{dy} \right) \Delta \left( \frac{d}{dx} \right) \prod_{i=1}^k f(x_i, y_i) &= \sum_{\mu} \text{sign}(\mu) \Delta \left( \frac{d}{dy} \right) \prod_{i=1}^k \frac{d^{\mu(i)-1}}{dx_i^{\mu(i)-1}} f(x_i, y_i) \\ &= \sum_{\mu} \text{sign}(\mu) \det_{k \times k} \left( \frac{d^{\mu(i)+j-2}}{dx_i^{\mu(i)-1} dy_i^{j-1}} f(x_i, y_i) \right) \end{aligned}$$

and so

$$\Delta \left( \frac{d}{dy} \right) \Delta \left( \frac{d}{dx} \right) \prod_{i=1}^k f(x_i, y_i) \Big|_{\substack{x_i=X, \\ y_i=Y}} = \sum_{\mu} \text{sign}(\mu) \det_{k \times k} \left( \frac{d^{\mu(i)+j-2}}{dX^{\mu(i)-1} dY^{j-1}} f(X, Y) \right).$$

Now, we may rearrange the rows of the matrix

$$\left( \frac{d^{\mu(i)+j-2}}{dX^{\mu(i)-1} dY^{j-1}} f(X, Y) \right)$$

to obtain

$$\left( \frac{d^{i+j-2}}{dX^{i-1} dY^{j-1}} f(X, Y) \right);$$

doing so cancels out the  $\text{sign}(\mu)$  attached to the determinant, and we reach the desired formula.  $\square$

Our final lemma is a recursion for determinants discovered by Lewis Carroll. It will be used in the proof of Theorem 4.

**Lemma 7.** *Let  $A$  be an  $k \times k$  matrix, and let  $A_{b_1, \dots, b_s}^{a_1, \dots, a_r}$  denote the matrix  $A$  with rows  $a_1, \dots, a_r$  and the columns  $b_1, \dots, b_s$  removed. Then*

$$\det A \begin{pmatrix} i \\ j \end{pmatrix} \cdot \det A \begin{pmatrix} j \\ i \end{pmatrix} - \det A \begin{pmatrix} i \\ j \end{pmatrix} \cdot \det A \begin{pmatrix} j \\ i \end{pmatrix} = \det A \cdot \det A \begin{pmatrix} i, j \\ i, j \end{pmatrix}.$$

### 3. PROOF OF THE THEOREMS

A simple calculation shows that

$$\begin{aligned} \frac{d^m}{d\alpha_1^m} \cdots \frac{d^m}{d\alpha_k^m} I(G(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) \Big|_{\alpha_1 = \dots = \alpha_k = 0} \\ = (-1)^{mk} \int_{G(2N)} \left( \sum_{j=0}^m \begin{Bmatrix} m \\ j \end{Bmatrix} \Lambda^{(j)}(1) \right)^k dA, \end{aligned}$$

where  $\begin{Bmatrix} m \\ j \end{Bmatrix}$  denotes a Stirling number of the second kind. It follows that

$$M_k(G(2N), m) = \frac{d^m}{d\alpha_1^m} \cdots \frac{d^m}{d\alpha_k^m} I(G(2N); e^{-\alpha_1}, \dots, e^{-\alpha_k}) \Big|_{\alpha_1 = \dots = \alpha_k = 0} (1 + O(N^{-1})).$$



Thus we may proceed towards Theorems 1, 2, and 3 by differentiating the formulae found in Corollaries 1, 2, and 3, respectively. The asymptotics remain valid after differentiating because they are uniform in  $\alpha_i$ .

*Proof of Theorem 1.* From Corollary 1 and the above argument, we know that the  $k$ th moment of  $\Lambda''$  is asymptotically

$$M_k(\mathrm{USp}(2N), 2) = \frac{(-1)^{k(k-1)/2}}{k!} \tilde{M}_k(\mathrm{USp}(2N), 2)(1 + O(N^{-1}))$$

where

$$\tilde{M}_k(\mathrm{USp}(2N), 2) = \frac{d^{2k}}{d\alpha_1^2 \cdots d\alpha_k^2} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{\Delta(w)\Delta(w^2)e^{N\sum_j(w_j - \alpha_j)}}{\prod_{i,j}(w_j^2 - \alpha_i^2)} dw_1 \cdots dw_k \Big|_{\alpha_1 \cdots \alpha_k = 0}.$$

We apply Lemma 4 with  $m = 2$ , finding

$$\begin{aligned} \frac{d^{2k}}{d\alpha_1^2 \cdots d\alpha_k^2} \frac{e^{-N\sum_{i=1}^k \alpha_i}}{\prod_{1 \leq i, j \leq k} (w_j^2 - \alpha_i^2)} \Big|_{\alpha_1 = \cdots = \alpha_k = 0} &= \left( \prod_{j=1}^k \frac{1}{w_j^{2k}} \right) \left( N^2 + 2 \sum_{j=1}^k \frac{1}{w_j^2} \right)^k \\ &= \frac{d^k}{dt^k} \left( \prod_{j=1}^k \frac{1}{w_j^{2k}} \right) \exp \left( tN^2 + 2t \sum_{j=1}^k \frac{1}{w_j^2} \right) \Big|_{t=0}. \end{aligned} \quad (6)$$

This allows us to write

$$\tilde{M}_k(\mathrm{USp}(2N), 2) = \frac{d^k}{dt^k} \frac{e^{tN^2}}{(2\pi i)^k} \oint \cdots \oint \Delta(w)\Delta(w^2) \exp \left( \sum_{j=1}^k Nw_j + \frac{2t}{w_j^2} \right) \frac{dw_1}{w_1^{2k}} \cdots \frac{dw_k}{w_k^{2k}} \Big|_{t=0}.$$

We now replace the Vandermonde determinants in this expression with Vandermonde determinants of differential operators. Observe that

$$\Delta(w^2) = \Delta \left( \frac{d}{dL} \right) e^{\sum_{i=1}^k w_i^2 L_i} \Big|_{L_i=0}.$$

and also

$$\Delta(w) \cdot e^{\sum_{i=1}^k Nw_i} = \Delta \left( \frac{d}{dM} \right) e^{\sum_{i=1}^k w_i M_i} \Big|_{M_i=N}$$

This implies that we can compute the integral in the above formula as

$$\begin{aligned}
& \frac{1}{(2\pi i)^k} \oint \cdots \oint \Delta(w) \Delta(w^2) \exp \left( \sum_{j=1}^k N w_j + \frac{2t}{w_j^2} \right) \frac{dw_1}{w_1^{2k}} \cdots \frac{dw_k}{w_k^{2k}} \\
&= \Delta \left( \frac{d}{dL} \right) \Delta \left( \frac{d}{dM} \right) \frac{1}{(2\pi i)^k} \oint \cdots \oint \exp \left( \sum_{j=1}^k L_j w_j^2 + M_j w_j + \frac{2t}{w_j^2} \right) \frac{dw_1}{w_1^{2k}} \cdots \frac{dw_k}{w_k^{2k}} \Big|_{\substack{L_j=0, \\ M_j=N}} \\
&= \Delta \left( \frac{d}{dL} \right) \Delta \left( \frac{d}{dM} \right) \prod_{j=1}^k \left( \frac{1}{2\pi i} \oint_{|w|=1} \exp \left( L_j w^2 + M_j w + \frac{2t}{w^2} \right) \frac{dw}{w^{2k}} \right) \Big|_{\substack{L_j=0, \\ M_j=N}} \\
&= k! \det_{k \times k} \left( \frac{d^{i+j-2}}{dL^{i-1} dM^{j-1}} \frac{1}{2\pi i} \oint_{|w|=1} \exp \left( L w^2 + M w + \frac{2t}{w^2} \right) \frac{dw}{w^{2k}} \Big|_{\substack{L=0, \\ M=N}} \right)
\end{aligned}$$

where the last equality is by Lemma 6. Using the fact that

$$\begin{aligned}
& \frac{d^{i+j-2}}{dL^{i-1} dM^{j-1}} \frac{1}{2\pi i} \oint_{|w|=1} \exp \left( L w^2 + M w + \frac{2t}{w^2} \right) \frac{dw}{w^{2k}} \Big|_{\substack{L=0, \\ M=N}} \\
&= \frac{1}{2\pi i} \oint \frac{\exp(Nw + \frac{2t}{w^2})}{w^{2k-2i-j+3}} dw \\
&= \frac{N^{2k-2i-j+2}}{2\pi i} \oint \frac{\exp(w + \frac{2tN^2}{w^2})}{w^{2k-2i-j+3}} dw,
\end{aligned}$$

we obtain a simplified formula for  $\tilde{M}_k(\mathrm{USp}(2N), 2)$ :

$$\begin{aligned}
\tilde{M}_k(\mathrm{USp}(2N), 2) &= k! N^{k(k+1)/2} \frac{d^k}{dt^k} \left( e^{tN^2} \det_{k \times k} \left( \frac{1}{2\pi i} \oint \frac{\exp(w + \frac{2tN^2}{w^2})}{w^{2k-2i-j+3}} dw \right) \right) \Big|_{t=0} \\
&= (-1)^{k(k-1)/2} k! N^{k(k+1)/2} \frac{d^k}{dt^k} \left( e^{tN^2} \det_{k \times k} \left( \frac{1}{2\pi i} \oint \frac{\exp(w + \frac{2tN^2}{w^2})}{w^{2i-j+1}} dw \right) \right) \Big|_{t=0} \\
&= (-1)^{k(k-1)/2} k! N^{k(k+5)/2} \frac{d^k}{du^k} \left( e^u \det_{k \times k} \left( \frac{1}{2\pi i} \oint \frac{\exp(w + \frac{2u}{w^2})}{w^{2i-j+1}} dw \right) \right) \Big|_{u=0}
\end{aligned}$$

where we have interchanged columns of the matrix to obtain the second line and set  $u = tN^2$  to obtain the third. The theorem follows.  $\square$

*Proof of Theorem 2.* We can proceed in the same way as in the proof of Theorem 1, starting from Corollary 2.  $\square$

*Proof of Theorem 3.* This time we consider the third derivative. From (6) we have

$$\begin{aligned}
\left. \frac{d^{3k}}{d\alpha_1^3 \cdots d\alpha_k^3} \frac{\prod_{i=1}^k \alpha_i e^{N \sum_{i=1}^k \alpha_i}}{\prod_{1 \leq i, j \leq k} (w_j^2 - \alpha_i^2)} \right|_{\alpha_1 = \cdots = \alpha_k = 0} &= 3 \frac{d^{2k}}{d\alpha_1^2 \cdots d\alpha_k^2} \frac{e^{N \sum_{i=1}^k \alpha_i}}{\prod_{1 \leq i, j \leq k} (w_j^2 - \alpha_i^2)} \Big|_{\alpha_1 = \cdots = \alpha_k = 0} \\
&= 3 \prod_{j=1}^k \frac{1}{w_j^{2k}} \left( N^2 + 2 \sum_{j=1}^k \frac{1}{w_j^2} \right)^k \\
&= 3 \frac{d^k}{dt^k} \prod_{j=1}^k \frac{1}{w_j^{2k}} \exp \left( t N^2 + 2t \sum_{j=1}^k \frac{1}{w_j^2} \right) \Big|_{t=0}
\end{aligned}$$

and the theorem follows from Corollary 3 in the same way.  $\square$

*Proof of Theorem 4.* We begin by proving a two-variable version of the recurrence relation. Let

$$\begin{aligned}
\tilde{g}_m(x, y) &:= \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{xz + \frac{y}{z^2}}}{z^{m+1}} dz, \\
\tilde{\mathcal{T}}_{k,\ell}(x, y) &:= \det_{k \times k} (\tilde{g}_{2i-j+\ell}(x, y)),
\end{aligned}$$

for  $x, y \in \mathbb{C}$ ,  $k \geq 0$  and  $\ell \in \mathbb{Z}$ . Using Lemma 7, we will show that:

$$\tilde{\mathcal{T}}_{k+1,\ell}(x, y) \tilde{\mathcal{T}}_{k-1,\ell}(x, y) = \tilde{\mathcal{T}}_{k,\ell}(x, y) \frac{\partial^2}{\partial x \partial y} \tilde{\mathcal{T}}_{k,\ell}(x, y) - \frac{\partial}{\partial x} \tilde{\mathcal{T}}_{k,\ell}(x, y) \frac{\partial}{\partial y} \tilde{\mathcal{T}}_{k,\ell}(x, y). \quad (7)$$

Let  $A_{k,\ell}$  denote the matrix of  $\tilde{\mathcal{T}}_{k,\ell}(x, y)$ , i.e.

$$A_{k,\ell} = \begin{pmatrix} \tilde{g}_{1+\ell} & \tilde{g}_\ell & \cdots & \tilde{g}_{2-k+\ell} \\ \tilde{g}_{3+\ell} & \tilde{g}_{2+\ell} & \cdots & \tilde{g}_{4-k+\ell} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{2k-1+\ell} & \tilde{g}_{2k-2+\ell} & \cdots & \tilde{g}_{k+\ell} \end{pmatrix}.$$

Observe that

$$\begin{aligned}
\frac{\partial}{\partial x} \tilde{g}_m(x, y) &= \tilde{g}_{m-1}(x, y, 2t), \\
\frac{\partial}{\partial y} \tilde{g}_m(x, y) &= \tilde{g}_{m+2}(x, y, 2t).
\end{aligned}$$

We now compute the partial derivatives of  $\tilde{\mathcal{T}}_{k,\ell}(x, y)$ . Expanding the derivative by columns, we obtain:

$$\begin{aligned}
\frac{\partial}{\partial x} \tilde{\mathcal{T}}_{k,\ell} &= \begin{vmatrix} \partial \tilde{g}_{1+\ell} / \partial x & \tilde{g}_\ell & \cdots & \tilde{g}_{2-k+\ell} \\ \partial \tilde{g}_{3+\ell} / \partial x & \tilde{g}_{2+\ell} & \cdots & \tilde{g}_{4-k+\ell} \\ \cdots & \cdots & \cdots & \cdots \\ \partial \tilde{g}_{2k-1+\ell} / \partial x & \tilde{g}_{2k-2+\ell} & \cdots & \tilde{g}_{k+\ell} \end{vmatrix} + \begin{vmatrix} \tilde{g}_{1+\ell} & \partial \tilde{g}_\ell / \partial x & \cdots & \tilde{g}_{2-k+\ell} \\ \tilde{g}_{3+\ell} & \partial \tilde{g}_{2+\ell} / \partial x & \cdots & \tilde{g}_{4-k+\ell} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{2k-1+\ell} & \partial \tilde{g}_{2k-1+\ell} / \partial x & \cdots & \tilde{g}_{k+\ell} \end{vmatrix} + \\
&\cdots + \begin{vmatrix} \tilde{g}_{1+\ell} & \tilde{g}_\ell & \cdots & \partial \tilde{g}_{2-k+\ell} / \partial x \\ \tilde{g}_{3+\ell} & \tilde{g}_{2+\ell} & \cdots & \partial \tilde{g}_{4-k+\ell} / \partial x \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{2k-1+\ell} & \tilde{g}_{2k-2+\ell} & \cdots & \partial \tilde{g}_{k+\ell} / \partial x \end{vmatrix}.
\end{aligned}$$

All terms but the last one in this sum vanish. Thus  $\frac{\partial}{\partial x} \tilde{\mathcal{T}}_{k,\ell} = \det A_{k+1,\ell} \binom{k+1}{k}$ , where  $A \binom{a_1, \dots, a_r}{b_1, \dots, b_s}$  denotes the matrix  $A$  with rows  $a_1, \dots, a_r$  and the columns  $b_1, \dots, b_s$  removed, as in Lemma 7.

Similarly, expanding the partial derivative by rows, we have  $\frac{\partial}{\partial y} \tilde{\mathcal{T}}_{k,\ell} = \det A_{k+1,\ell} \binom{k}{k+1}$ . Finally,  $\frac{\partial^2}{\partial x \partial y} \tilde{\mathcal{T}}_{k,\ell} = \det A_{k+1,\ell} \binom{k}{k}$ . Thus Equation 7 is an immediate consequence of Lemma 7.

To return to our original functions  $g_m(u)$  and  $\mathcal{T}_{k,\ell}(u)$ , we observe that a simple change of variables gives  $\tilde{g}(x, y) = x^m g_m(x^2 y)$  for any  $x \neq 0$ . Further, removing a factor of  $x^{2i+\ell}$  from each row  $i$  and  $x^{-j}$  from each column  $j$  in the determinant, we have

$$\tilde{\mathcal{T}}_{k,\ell}(x, y) = x^{\frac{k(k+1)}{2} + k\ell} \mathcal{T}_{k,\ell}(x^2 y). \quad (8)$$

If we set  $u = x^2 y$ , then by the chain rule we have  $\frac{\partial}{\partial x} = 2xy \frac{d}{du}$ ,  $\frac{\partial}{\partial y} = x^2 \frac{d}{du}$ , and so

$$\begin{aligned} & x^{\frac{k(k+1)}{2} + k\ell} \mathcal{T}_{k,\ell}(u) \frac{\partial^2}{\partial x \partial y} \left( x^{\frac{k(k+1)}{2} + k\ell} \mathcal{T}_{k,\ell}(u) \right) \\ &= x^{k(k+1) + 2k\ell + 1} \left( \left( \frac{k(k+1)}{2} + k\ell + 2 \right) \mathcal{T}_{k,\ell}(u) \mathcal{T}'_{k,\ell}(u) + 2u \mathcal{T}_{k,\ell}(u) \mathcal{T}''_{k,\ell}(u) \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial}{\partial x} \left( x^{\frac{k(k+1)}{2} + k\ell} \mathcal{T}_{k,\ell}(u) \right) \frac{\partial}{\partial y} \left( x^{\frac{k(k+1)}{2} + k\ell} \mathcal{T}_{k,\ell}(u) \right) \\ &= x^{k(k+1) + 2k\ell + 1} \left( \left( \frac{k(k+1)}{2} + k\ell \right) \mathcal{T}_{k,\ell}(u) \mathcal{T}'_{k,\ell}(u) + 2u \mathcal{T}'_{k,\ell}(u)^2 \right). \end{aligned}$$

Thus the theorem follows from (8) and (7). □

#### 4. NUMERICAL VALUES

Below are the first several values for the constant  $b_k(\mathrm{USp}(2N, 2))$ . See Theorem 1.

$$\begin{aligned} b_1 &= \frac{1}{2 \cdot 3} \\ b_2 &= \frac{19}{2^4 \cdot 3^2 \cdot 5 \cdot 7} \\ b_3 &= \frac{487}{2^7 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11} \\ b_4 &= \frac{59 \cdot 197}{2^{13} \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} \\ b_5 &= \frac{174290791}{2^{19} \cdot 3^{10} \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19} \\ b_6 &= \frac{3373 \cdot 1670407}{2^{25} \cdot 3^{14} \cdot 5^6 \cdot 7^3 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23} \\ b_7 &= \frac{37 \cdot 83 \cdot 2203 \cdot 3571457}{2^{32} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19 \cdot 23} \\ b_8 &= \frac{61 \cdot 595351 \cdot 11423948521}{2^{42} \cdot 3^{23} \cdot 5^{11} \cdot 7^7 \cdot 11^4 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31} \end{aligned}$$

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$$b_9 = \frac{53 \cdot 16646765854629827113}{2^{53} \cdot 3^{29} \cdot 5^{13} \cdot 7^9 \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31}$$

$$b_{10} = \frac{47 \cdot 1553 \cdot 1787 \cdot 73709 \cdot 152825093}{2^{62} \cdot 3^{34} \cdot 5^{17} \cdot 7^{10} \cdot 11^5 \cdot 13^5 \cdot 17^4 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37}$$

Below are several values for the constant  $b_k(\mathrm{SO}(2N), 2)$ . See Theorem 2.

$$b_1 = 1$$

$$b_2 = \frac{7}{2 \cdot 3 \cdot 5}$$

$$b_3 = \frac{2 \cdot 13}{3^4 \cdot 5 \cdot 7}$$

$$b_4 = \frac{17 \cdot 5987}{2^6 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}$$

$$b_5 = \frac{157 \cdot 17519}{2^9 \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17}$$

$$b_6 = \frac{22273664659}{2^{15} \cdot 3^{12} \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19}$$

$$b_7 = \frac{116228886131}{2^{14} \cdot 3^{14} \cdot 5^8 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23}$$

$$b_8 = \frac{36774351481263481}{2^{28} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^4 \cdot 13^3 \cdot 17 \cdot 19^2 \cdot 23 \cdot 29}$$

$$b_9 = \frac{71 \cdot 103 \cdot 223 \cdot 661 \cdot 1069 \cdot 134437}{2^{34} \cdot 3^{25} \cdot 5^{11} \cdot 7^6 \cdot 11^4 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29}$$

$$b_{10} = \frac{25171 \cdot 7695491 \cdot 57668937071891}{2^{45} \cdot 3^{29} \cdot 5^{15} \cdot 7^9 \cdot 11^5 \cdot 13^5 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37}$$

We omit a table of values for the odd orthogonal case. Recall that by Theorems 1 and 3,  $b_k(\mathrm{O}^-(2N), 3) = 3 \cdot 2^k \cdot b_k(\mathrm{USp}(2N), 2)$  so that these values are given in terms of the above table for the symplectic case.

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S. ALI ALTUĞ – COLUMBIA UNIVERSITY, DEPARTMENT OF MATHEMATICS, ROOM 517, 2990 BROADWAY, NEW YORK, NY 10027, USA

SANDRO BETTIN – CENTRE DE RECHERCHES MATHÉMATIQUES - UNIVERSITÉ DE MONTRÉAL, P.O. BOX 6128, CENTRE-VILLE STATION, MONTRÉAL, QC, H3C 3J7, CANADA

IAN PETROW – STANFORD UNIVERSITY, DEPARTMENT OF MATHEMATICS, 450 SERRA MALL, BUILDING #380, STANFORD, CA, 94305, USA

RISHIKESH – CONCORDIA UNIVERSITY, 1455 DE MAISONNEUVE BLVD. WEST, LB-901, MONTRÉAL, QC, H3G 1M8, CANADA

IAN WHITEHEAD – COLUMBIA UNIVERSITY, DEPARTMENT OF MATHEMATICS, ROOM 610, 2990 BROADWAY, NEW YORK, NY 10027, USA